Response of a nonlinear Hamiltonian system to the external harmonic field: Resonant and chaotic cases

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The response of a classical nonlinear oscillator to the homogeneous external field with harmonic time dependence is studied beyond the domain of applicability of the perturbation theory. The new quantity, a harmonic susceptibility (HS) that is proportional to the ensemble average of the Fourier amplitude of motion with the same frequency and phase as the external field, is introduced. In the off-resonant region, HS tends asymptotically to usual linear susceptibility. The cases of the intermediate field strength, when the isolated nonlinear resonance prevails, and that of the strong field, when the resonances overlap and the extended chaotic component is formed in the phase space, are studied. For both cases the analytical expressions for HS are obtained in the form of quadratures and confirmed by the comparison with the results of direct numerical simulation. [S1063-651X(99)06812-9]

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I. INTRODUCTION

In this paper we study the response of a classical Hamiltonian system to the external field with a harmonic dependence on time and medium or strong magnitudes. This model can describe the micro-objects (such as atoms, clusters, molecules, and their ions) in highly excited states with large principal quantum number $n \gg 1$, under the influence of the external monochromatic electromagnetic wave. In these cases the value of Planck's constant in natural units of the system is small, $\hbar \approx n^{-1} \ll 1$, and the radiation damping is negligible.

Let us assume that the unperturbed system can be described by an autonomous conservative model with the Hamiltonian function

$$H_0(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2M} + U(\vec{r}), \tag{1.1}$$

where $\vec{r}$ and $\vec{p}$ are vectors of the Cartesian coordinates and the components of momentum correspondingly, and M is the mass of the particle. The external perturbation will be described by the addition to H_0 the term

$$V(x,t) = -xF\cos\omega t, \tag{1.2}$$

where the *active coordinate x* is one of the Cartesian coordinates, and the scalar quantity F for obvious reasons will be called the external field.

The theory of response is constructed for the ensembles E of the systems, in which the average value of the active coordinate in the absence of the external field (for $F=0$) equals zero, [1], (p. 421):

$$\langle x(t)_0 \rangle_E = 0. \tag{1.3}$$

For classical models, the ensemble E is usually taken as a microcanonical one ($H_0 = E$) with the uniform (on the induced measure) distribution of the phase density on the invariant component of the phase space [2].

By definition, the response of the system to the external field is given by the ensemble average of the law of motion of the active coordinate of the perturbed system:

$$R(t;F,\omega) = \langle x(t) \rangle_E. \tag{1.4}$$

In the limit of infinitely small external field magnitude, the response can be written in the form

$$R(t;F,\omega) = F[\alpha'(\omega)\cos\omega t + \alpha''(\omega)\sin\omega t], \tag{1.5}$$

where the quantities α' and α'' are the real and imaginary parts of the *linear susceptibility*. The explicit form of linear susceptibility, given by nonstationary quantum perturbation theory, is well known [1], [p. 440],

$$\alpha(\omega) = \frac{1}{\hbar}\sum_k \frac{2|x_{nk}|^2\omega_{kn}}{\omega_{kn}^2 - (\omega + i0)^2}, \tag{1.6}$$

where x_{nk} is the matrix element of the active coordinate, ω_{kn} is a transition frequency between the states $|k\rangle$ and $|n\rangle$, and $\omega_{kn} = (E_k - E_n)/\hbar$.

Its classical analog for one-dimensional systems was obtained in [3]

$$\alpha(\omega) = 2\Omega\sum_{s=1}^{\infty} \frac{d}{dE}\left(\frac{s^2|X_s|^2\Omega}{s^2\Omega^2 + (\omega + i0)^2}\right), \tag{1.7}$$

where X_s is the amplitude of the sth Fourier harmonic of the unperturbed motion, and Ω is the main harmonic frequency.

The linear susceptibility is well defined only in the limit of the infinitely small field magnitude, when the response can be written in the form (1.5). For finite although arbitrarily small values of F, the linear susceptibility loses validity in the resonant case, that is, for sufficiently small frequency mismatches. In this domain of parameters the

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external field strongly influences the motion of the system. To describe the system's response in this case the methods that go beyond the perturbation theory are needed.

Nonperturbative approaches to the problem of the response have been developed in the frame of the quantum theory. For strongly anharmonic systems near the resonance the model of a two-level system can be used [4], whereas for weakly anharmonic ones the model of quantum nonlinear resonance in the rotating-wave approximation [5–7] could be applied. However, the applicability of these methods is controlled by the conditions of smallness of certain dimensionless parameters that are proportional to $\hbar^{-1}$. Therefore it is impossible to stretch these approaches to the quasiclassical limit $\hbar \to 0$.

To describe the response of a classical system to the resonant field, we shall introduce a new quantity, the harmonic susceptibility, which extends the notion of the linear susceptibility to arbitrary values of amplitude and frequency of the external field, and whose definition will be valid irrespective of the character of the systems motion, whether it is regular or chaotic.

If for $t<0$ the ensemble E of the systems (1.1) was in a state that had the property (1.3), and if for $t>0$ the field was turned on within the time interval $0<t<\tau$ so that for $t>\tau$ the field can be described by the model (1.2), then the quantity

$$A(\omega)=\lim_{T\to\infty}\frac{2}{FT}\int_0^T\langle x(t)\rangle\cos\omega t dt \tag{1.8}$$

will be called the *harmonic susceptibility* of the system. This quantity is proportional to the amplitude of the Fourier component of the averaged perturbed motion of the system, which coincides with the external field in frequency and in phase. The harmonic susceptibility depends in general on the field strength (see the Appendix) and in the limit $F\to 0$ is in agreement with the linear susceptibility: if the response of the system has the form (1.5), then the harmonic susceptibility

$$A(\omega)=\alpha'(\omega). \tag{1.9}$$

The purpose of this paper is to study the properties of the harmonic susceptibility of a nonlinear system with one degree of freedom in the cases of regular and chaotic motion of the system. In Sec. II the model is described and the pendulum Hamiltonian is derived. In Sec. III the harmonic susceptibility is found for an intermediate field magnitude, when only one nonlinear resonance dominates. In Sec. IV the harmonic susceptibility is found for a strong field, when widespread chaos is observed. The main results are summarized in Sec. V.

II. THE MODEL

As an example for the study we took the Duffing oscillator with the two-well potential, that is, model (1.1) and (1.2) with the Hamiltonian function

$$H_0(x,p)=\frac{p^2}{2M}-Ax^2+Bx^4. \tag{2.1}$$

In the following we use the system of units in which $M=1$, $A=1/2$, and $B=1/4$.

We shall restrict ourselves by studying the energy range $E>0$, where there is only one connected domain of the classical motion, and introduce the action-angle variables for the unperturbed system. The action variable I is given by the expression

$$I(E)=\frac{1}{2\pi}\oint p\,dx=\frac{2\sqrt{\varepsilon}}{3\pi}[(\varepsilon-1)K(\varepsilon_+)+2K(\varepsilon_-)], \tag{2.2}$$

where $\varepsilon=\sqrt{1+4E}$,

$$\varepsilon_+=\sqrt{\frac{\varepsilon+1}{2\varepsilon}},\quad \varepsilon_-=\sqrt{\frac{\varepsilon-1}{2\varepsilon}}, \tag{2.3}$$

and $K(z)$ is the complete elliptic integral of the first kind. Equation (2.2) defines implicitly the Hamiltonian function $H_0(I)=E$ of the unperturbed two-well oscillator. Further, we treat the dependence of variables on the energy E and on the action I on equal footing. The frequency of the unperturbed motion $\Omega(I)$, which yields the rate of increase of the angle variable θ, is given by the relation

$$\Omega(I)=\frac{dH_0(I)}{dI}=\frac{\pi\sqrt{\varepsilon}}{2K(\varepsilon_+)}. \tag{2.4}$$

For a given energy value E [or the corresponding action value $I(E)$] and the initial condition $x(0)$, the law of motion $x(t)$ can be expressed explicitly in terms of the Jacobi elliptic functions. This motion is periodic and can be represented by its Fourier expansion:

$$x(t)=\sum_{n=1}^{\infty}a_{2n-1}(I)\cos[(2n-1)\theta], \tag{2.5}$$

where $\theta=\Omega(I)(t-t')$, and t' is the moment of time when the particle is at the right turning point.

The Fourier amplitudes $a_{2n-1}(I)$ are given by

$$a_{2n-1}(I)=\frac{\pi\sqrt{2\varepsilon}}{K(\varepsilon_+)}\cosh^{-1}\left((2n-1)\pi\frac{K(\varepsilon_-)}{K(\varepsilon_+)}\right). \tag{2.6}$$

At a given value of I, the amplitudes decrease exponentially with the growth of n. If the value of the energy E is not too small $(E>1)$, then the first harmonic dominates in the law of motion: $a_{2n-1}/a_1\ll 1$ for all $n\geqslant 2$. Thus the one-mode approximation of the law of motion,

$$x(t)\approx a_1(I)\cos\theta, \tag{2.7}$$

can be used in this energy range.

The perturbed motion of the system can be studied in the paradigm of the nonlinear resonance [8–10]. Let us consider the system (2.1) with the initial energy E under the external perturbation (1.2), which is suddenly imposed $(\tau=0)$ at the moment $t=0$. With the assumption (2.7), the Hamiltonian function of the perturbed system can be written in the form

$$H(I,\theta,t)=H_0(I)-Fa_1(I)\cos\theta\cos\omega t. \tag{2.8}$$

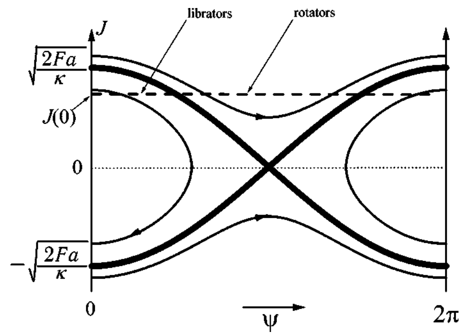

FIG. 1. The localization of ensemble P (see Sec. III A) in the phase space of the pendulum (2.13) (dimensionless units). The initial angular momentum $J(0)$ is the same for all the pendulums; the initial phase $\psi(0)$ is distributed uniformly within $[0,2\pi]$. The trajectories outside the separatrix are rotators, the trajectories within the separatrix are librators. Librators exist only if $|J(0)| < \sqrt{2Fa/\kappa}$.

The corresponding equations of motion are

$$\dot{I} = -Fa_1 \sin\theta \cos\omega t, \quad \dot{\theta} = \Omega(I) - F\frac{da_1}{dI}\cos\theta\cos\omega t. \tag{2.9}$$

Let I_0 denote the value of the action variable of the unperturbed model such that the corresponding frequency equals that of perturbation, $\Omega(I_0)=\omega$. If the field frequency is close to that of the unperturbed motion, $\omega \approx \Omega(E)$, then the frequency dependence on the action can be linearized:

$$\Omega(I) \approx \Omega(I_0) + \kappa J, \tag{2.10}$$

where the nonlinearity parameter κ is equal to the derivative of the frequency on action,

$$\kappa = \frac{d\Omega(I)}{dI}, \tag{2.11}$$

taken at the value $I=I_0$, and $J=I-I_0$ is the action mismatch assumed to be small. By the introduction of the slow angle variable $\psi=\theta-\omega t$, neglecting all nonresonant terms and ignoring the dependence of the first harmonic amplitude on the action value, $a_1(I)\approx a_1(I_0)\equiv a$, Eqs. (2.9) turn into the system

$$\dot{J} \approx -\frac{Fa}{2}\sin\psi, \quad \dot{\psi} = \kappa J. \tag{2.12}$$

These equations are the canonical ones for the Hamiltonian function of the pendulum,

$$H_p(J,\psi) = \frac{\kappa}{2}J^2 - \frac{Fa}{2}\cos\psi. \tag{2.13}$$

The variable J is the angular momentum of the pendulum and ψ is the angle variable. The transition from the Hamiltonian (2.8) to the effective Hamiltonian (2.13) is a standard procedure of the nonlinear dynamics, discussed in many sources [9,10].

The phase space of the pendulum (see Fig. 1) has three distinct regions. Within the separatrix of the system, for $|J| < \sqrt{2Fa/\kappa}\sin(\psi/2)$, the phase points are captured in the nonlinear resonance and the motion is librating $(\langle\dot{\psi}\rangle_t \neq 0)$. Outside the separatrix, where $|J| > \sqrt{2Fa/\kappa}\sin(\psi/2)$, the motion of the pendulum is rotating $(\langle\dot{\psi}\rangle_t = 0)$. The frequency halfwidth of the nonlinear resonance Δ_r at the energy value $E = E(I_0)$ can be defined from Eqs. (2.13) and (2.10) as

$$\Delta_r = \sqrt{2Fa\kappa}. \tag{2.14}$$

The pendulum Hamiltonian is valid for describing the motion of system (2.1) if the higher terms of the functions $\Omega(I)$ and $a_1(I)$ are negligible at all possible values of action mismatch within the separatrix. These conditions impose the limitations (from above) on the field strength, which, however, was never violated in the physically interesting range of parameters values. The other limitations of the single resonance pendulum model are related to a possible overlap of resonances; they will be discussed in Sec. III C. We note in passing that the resonance model remains valid however small the frequency mismatch may be.

III. THE RESONANT CASE

A. The analytical solution

In this section we shall study the response of the system in the domain of parameters where the pendulum model with the single resonance is applicable. Let us consider the microcanonical ensemble E of two-well oscillators with the same energy E (and, consequently, the same value of action I) and random phases $\theta(t)$ at any given moment of time. If the perturbation is switched on instantly at $t=0$, then at $t=+0$ the ensemble P of pendulums which corresponds to the ensemble E consists of systems with the same value of the angular momentum $J(0)=I(E)-I_0=\Delta/\kappa$, where $\Delta = \Omega(E)-\omega$ is the frequency mismatch, and with initial phases $\psi(0)=\varphi$, distributed uniformly within $[0,2\pi]$ (see Fig. 1). The effective energies $E_p = H_p(J,\psi)$ of these pendulums, consequently, are different. They belong to the interval

$$\frac{\kappa}{2}J^2(0) - \frac{Fa}{2} \leqslant E_p \leqslant \frac{\kappa}{2}J^2(0) + \frac{Fa}{2}. \tag{3.1}$$

If the modulus of the frequency mismatch does not exceed the frequency half-width of the resonance, $|\Delta| \leqslant \Delta_r$, then some phase points of the ensemble of pendulums P are captured in the nonlinear resonance (librators) and some phase points stay outside the separatrix (rotators). If $|\Delta| > \Delta_r$, then all points of the ensemble stay outside the separatrix and only rotation is possible.

In the one-mode approximation (2.7), the law of motion $x(t)$ can be written as

$$x(t) \approx a_1(t,\varphi)\cos\psi(t,\varphi)\cos\omega t. \tag{3.2}$$

From the definition of the harmonic susceptibility (1.8) for the ensemble E, we have

$$A(\omega) = \frac{a}{F}\langle\langle\cos\psi(t)\rangle_t\rangle_P, \tag{3.3}$$

where $\langle\ \rangle_t$ denotes the averaging over the period of pendulum (which depends on its initial phase φ), and $\langle\ \rangle_P$ denotes the ensemble averaging (over φ).

The time averaging in Eq. (3.3) can be carried out analytically. Let us introduce the dimensionless function

$$\epsilon(\varphi)=\frac{2}{Fa}H_p(J(0),\varphi)=\zeta-\cos\varphi, \qquad (3.4)$$

where $\zeta=\kappa J(0)^2/Fa$. Then for the rotator states $(-1<\epsilon<1)$

$$\langle\cos\psi(t,\varphi)\rangle_t^{(r)}=-1+2\frac{E(\eta)}{K(\eta)}, \quad \eta=\sqrt{\frac{1+\epsilon}{2}}, \qquad (3.5)$$

and for the librator states $(\epsilon>1)$

$$\langle\cos\psi(t,\varphi)\rangle_t^{(l)}=-\alpha+(\alpha+1)\frac{E(\xi)}{K(\xi)}, \quad \xi=\sqrt{\frac{2}{1+\epsilon}}. \qquad (3.6)$$

Here $K(z)$ and $E(z)$ are the complete elliptic integrals of the first and second kinds correspondingly. Thus the harmonic susceptibility of the microcanonical ensemble of Duffing oscillators with the energy E in the 1:1 resonance is given by the following formula:

$$A(\omega)=\frac{a}{2\pi F}\left[\int_{-\beta}^{\beta}\langle\cos\psi(t,\varphi)\rangle_t^{(r)}d\varphi + \int_{\beta}^{2\pi-\beta}\langle\cos\psi(t,\varphi)\rangle_t^{(l)}d\varphi\right], \qquad (3.7)$$

where $\beta=\arccos(\zeta-1)$ for $0<\zeta<2$ and $\beta=0$ for $\zeta>2$.

Equation (3.7) permits us to segregate the asymptotic forms of harmonic susceptibility. For large frequency mismatches $(\Delta\gg\Delta_r)$, the harmonic susceptibility in the resonant approximation has the asymptotic form

$$A(\omega)\approx-\frac{\kappa a^2}{2(\Omega-\omega)^2}. \qquad (3.8)$$

In this nonresonant case the perturbative approach is valid; it gives for the linear susceptibility the same asymptotic (3.8). It means that Eq. (3.7) can be used for the calculation of the harmonic susceptibility of a nonlinear system far away from the resonance as long as the one-resonance approximation holds.

B. The numerical calculations

In the derivation of the pendulum model (2.13), several approximations have been made that have unclear influence on the numerical accuracy of the final result. Therefore, it is worthwhile to compare the theoretical results with the direct numerical simulation of the motion of the system.

This comparison is presented in the Fig. 2. The theoretical and experimental values of the harmonic susceptibility A are plotted as functions of the perturbation frequency ω for the ensemble E with the initial energy $E=1$ for the field value $F=4.24\times10^{-2}$.

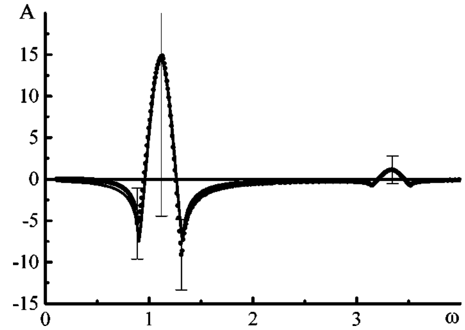

FIG. 2. The harmonic susceptibility $A(\omega)$ of the Duffing oscillator in the resonant case for values of the initial energy $E=1$ and the field amplitude $F=4.24\times10^{-2}$. All the units are dimensionless. Solid line, calculations based on the formula (3.7); circles, the result of numerical simulations. The bars indicate the standard deviation δA in the ensemble. The resonance at the third harmonic is clearly visible in the vicinity of $\omega\approx3.3$ (see Sec. III C).

To obtain one point of the theoretical plot, the integrals (3.7) were calculated over the total number of points $N=10^2$. The numbers of points $N^{(r)}$ and $N^{(l)}$ used for the calculation of the contributions of rotators and librators in the harmonic susceptibility depended on the value of β: $N^{(r)}=[N\beta/\pi]$, $N^{(l)}=N-N^{(r)}$, where rectangular brackets [ ] denote the truncation of a fractional part of a number in the brackets.

In the numerical experiment, the ensemble E was formed by 10^2 different trajectories distributed uniformly along the line, which corresponded to the same energy $E=1$. For each set of the initial values $\{x_i,p_i\}$, the equations of motion were integrated numerically by the Runge-Kutta Fourth-order method with the time step $\Delta t=\min\{0.05,2\pi/50\omega^{-1}\}$. Simultaneously with the integration, the contribution $A_i(\omega)$ of the trajectory to the harmonic susceptibility was calculated as

$$A_i(\omega)=\sum_k x_i(t_k)\cos(\omega t_k), \qquad (3.9)$$

where t_k is the moment of time, which corresponds to the kth integration step, $t_k=k\Delta t$. The time averaging was carried over the intervals $T=2\pi\times10^3\omega^{-1}$. Finally, the average value $A(\omega)$ and the standard deviation $\delta A(\omega)$ of the array $\{A_i(\omega)\}$ were calculated.

The most prominent feature of the dependence $A(\omega)$ is a maximum at the center of the resonance. The estimate of the upper limit of the maximal value of $A(\omega)$ in the exact resonance is given by a/F, which for our set of parameters yields $\max A<40$. The rotators always produce negative contributions to $A(\omega)$; they dominate when the frequency mismatch exceeds Δ_r. Outside the resonant domain the behavior of $A(\omega)$ is close to that of the perturbative linear susceptibility given by Eq. (1.7); the negative contribution from the term of the second order in Δ^{-1} prevails everywhere in the nonresonant domain $|\Delta|>\Delta_r$ except the range of low frequencies: $A(\omega)>0$ if $\omega<0.32$. The absolute value of harmonic susceptibility is too small in this range to be visible in the plot.

The dependence of the form of $A(\omega)$ on the parameters is clear. The variation of the initial energy E shifts the curve as a whole reflecting the change of the first harmonic frequency. The increase of the field amplitude F decreases the height of the resonant maximum and broadens it. On the contrary, if $F\rightarrow 0$, then the resonance width diminishes like $\Delta_r \sim \sqrt{F}$, and the peak value increases like $A_+ \sim 1/F$.

C. The higher resonances

The numerical results presented in Fig. 2 show also a smaller peak of the harmonic susceptibility at the frequency $\omega \approx 3.3$. It can be interpreted as a trace of the resonance between the frequency of the third harmonic 3Ω of the unperturbed motion of the system and the frequency of the external field ω [following the ratio $\Omega(E)/\omega$, we shall call it subharmonic resonance 1:3]. To study this region, the new pendulum model must be constructed in the way described in Sec. II.

By taking into account only the resonant third harmonic of the unperturbed motion, we come to the effective pendulum Hamiltonian,

$$\tilde{H}_p(\tilde{J},\tilde{\psi}) = \frac{3\tilde{\kappa}}{2}\tilde{J}^2 - \frac{F\tilde{a}}{2}\cos\tilde{\psi}, \tag{3.10}$$

where

$$\tilde{\kappa} = \frac{d\Omega(I)}{dI}, \quad \tilde{a} = a_3(I), \tag{3.11}$$

taken at $I=\tilde{I}_0$, which is defined by the equation $3\Omega(\tilde{I}_0) = \omega$; $\tilde{J}=I-\tilde{I}_0$, $\tilde{\psi}=3\theta-\omega t$. From Eq. (3.10), the maximal mismatch between the third harmonic of the unperturbed motion and the field frequency, which we shall call the frequency half-width of 1:3 resonance at the energy $E = E(\tilde{I}_0)$, $\tilde{\Delta}_r$, is equal to

$$\tilde{\Delta}_r = 3\sqrt{2F\tilde{a}\tilde{\kappa}}. \tag{3.12}$$

The widths of 1:1 and 1:3 resonances, which correspond to the same value of energy E, are related as

$$\frac{\tilde{\Delta}_r}{\Delta_r} = 3\sqrt{\frac{\tilde{a}}{a}}, \tag{3.13}$$

since $I_0=\tilde{I}_0$ in this case. For energy $E=1$ the ratio $\tilde{\Delta}_r/\Delta_r \approx 0.88$, which is in good agreement with Fig. 2.

IV. THE CHAOTIC CASE

In the preceding section we assumed that the perturbed ensemble remains always near one of the isolated nonlinear resonances. In this case the motion of each system is quasiperiodic with two frequencies, one of which coincides with the frequency of the perturbation, and another depends on the initial conditions.

The growth of F leads to the increase of the widths of the resonances [see Eq. (2.14)]. For every pair of resonances there is a critical value F_χ of field strength at which they overlap. This event is usually treated as a cause of the appearance of widespread chaos (Chirikov's criterion) [9,10]. The value F_χ is called ''chaos threshold'' and gives an estimate of a perturbation strength that leads to the widespread chaotic motion.

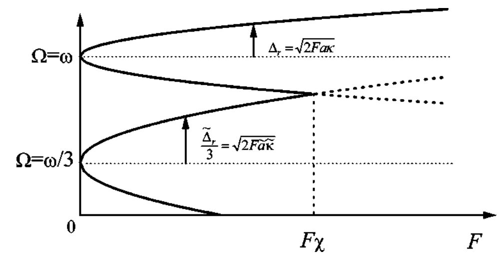

FIG. 3. The overlap of 1:1 and 1:3 resonances (schematically; arbitrary units). The upper and lower borders of the 1:1 and 1:3 resonances are plotted by solid lines. The widths of the resonances increase with the field magnitude. The overlap of the resonances that occurs at F_χ indicates a threshold of the widespread chaos.

For the Duffing oscillator in the range of initial energies $E\sim 1$, the chaos threshold F_χ for the given frequency of the perturbation is governed by the overlap of 1:1 and 1:3 resonances (see Fig. 3): the widespread chaos appears when the field of frequency ω and amplitude F becomes resonant for both the first harmonic of the unperturbed motion at the energy E and for the third harmonic of the unperturbed motion at a smaller value of energy $\tilde{E}$. The condition of overlap of resonances can be written in the form

$$\omega - \Delta_r = \frac{\omega}{3} + \frac{\tilde{\Delta}_r}{3}. \tag{4.1}$$

From Eqs. (2.14), (3.12), and (4.1) we obtain for the chaos threshold

$$F_\chi = \frac{4\omega^2}{9(\sqrt{2\kappa a} + \sqrt{2\tilde{\kappa}\tilde{a}})^2}. \tag{4.2}$$

In Fig. 4 the chaos threshold F_χ is plotted as a function of field frequency ω. Figure 4 shows that the value of the field $F=1.0$ provides the widespread chaos in the range of field frequencies $\omega < 2.07$.

Although the chaotic motion is mixing, thus leading to the continuous power spectra of dynamic variables, the harmonic susceptibility of the ensemble does not vanish. For the system that is strongly perturbed by the harmonic external field, the spectrum contains also a singular component at the

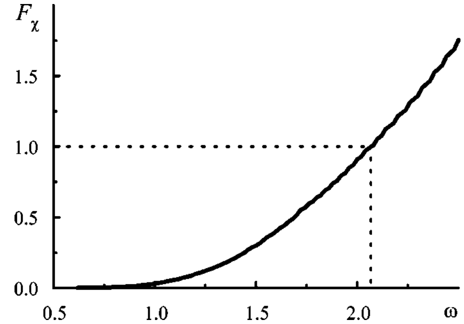

FIG. 4. Chaos threshold F_χ as a function of field frequency ω (dimensionless units). The external periodic field with the amplitude $F=1$ leads to chaotic motion, if its frequency satisfies the inequality $\omega < 2.07$.

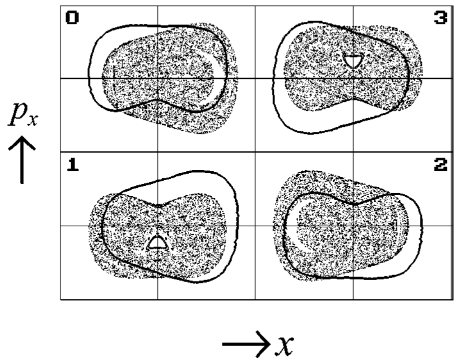

FIG. 5. Traces of the phase trajectory of a strongly perturbed system on the Poincaré sections (OY, momentum of particle; OX, its coordinate; units are dimensionless) for the ensemble with $E=1$. The stroboscopic sections $\{j\}$ represent the moments of time $t_k=(2\pi/\omega)(k+j/4)$. Lines denote the external border of the section taken at the previous value of phase.

frequency ω. Thus the motion of the system has the properties of a nonstationary random process. This is clearly seen from the traces of the phase trajectory on the Poincaré sections (see Fig. 5) that are taken at different values of the phase ϕ of the perturbing field. Patterns on the Poincaré sections demonstrate that for the chosen values $F=1.0$, $E=1.0$, and $\omega=1.0$ the chaotic component of the phase space covers nearly all the surface of the section. On the other hand, the shape and the position of the section depend on the field phase ϕ periodically, i.e., regularly. Hence, chaotic and regular responses coexist.

For the numerical calculation of the harmonic susceptibility $A(\omega)$ in the chaotic regime we have taken values $F=1.0$, $E=1.0$. The dependence is shown in Fig. 6 by the thin line. We note the following specific features: first, there is a plateau at the low field frequencies $\omega<\Omega(E)$; second, the resonance shape is qualitatively reproduced in the range $\omega>\Omega(E)$. The origins of both of these features are clear. Given the field magnitude $F=1.0$, in the case of high-frequency perturbation $\omega>2.07$, the motion remains regular since the 1:1 resonance persists. In the low-frequency range, $\omega<2.07$, the motion is chaotic. Since chaos originates from the overlap of 1:1 resonance at energy E with $1:n$ subresonances, which correspond to smaller values of energy, the

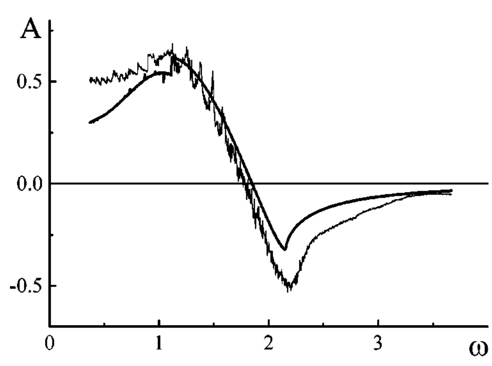

FIG. 6. The harmonic susceptibility $A(\omega)$ of the Duffing oscillator in the chaotic case for values of the initial energy $E=1$ and the field amplitude $F=1$. All the units are dimensionless. Solid line, calculations based on the formulas (4.3) and (4.4); dots, the result of numerical simulations.

initially microcanonical ensemble E of Duffing oscillators diffuses to lower energies within the time of the order of the inverse Lyapunov exponent. Thus, the effective ensemble E', which actually responds on the perturbation, essentially differs from E. We can think about every system in the ensemble as if it were in the 1:1 resonance with the external field. This assumption is justified by the great width of this resonance. The positive contribution of systems with small frequency mismatches to the harmonic susceptibility prevails and leads to the plateau in the dependence $A(\omega)$.

To make the analytical description of harmonic susceptibility of the effective ensemble E' tractable, we use a set of simplifying assumptions. First, only the main (1:1) resonances are taken into account and considered to be isolated. Second, we assume that the energy of a particle is changing by rare jumps. Third, we assume that in the domain $\omega<\Omega(E)$ the effective ensemble E' has the uniform energy distribution. If $\omega>\Omega(E)$, then the effective ensemble E' does not form and the response can be described as a resonant one.

With these assumptions the harmonic susceptibility can be cast in the form of the onefold or twofold quadratures:

$$A(\omega)=A^{\mathrm{res}}(\omega;E) \quad \text{for} \quad \omega>\Omega(E) \tag{4.3}$$

and

$$A(\omega)=\int_0^E A^{\mathrm{res}}(\omega;\varepsilon)\frac{d\varepsilon}{E} \quad \text{for} \quad \omega<\Omega(E), \tag{4.4}$$

where $A^{\mathrm{res}}(\omega;\varepsilon)$ denotes the resonant susceptibility of the microcanonical ensemble of Duffing oscillators with the energy ε given by Eq. (3.7). In Fig. 6 the result of the numerical calculation is shown as a solid line. The agreement is quite satisfactory, especially if one takes into account the crudeness of the simplifying approximations.

The theory presented above uses the concept of the instantaneous frequency of motion $\Omega(E)$, which is appropriate only if $\dot{\Omega}(2\pi/\Omega)\ll\Omega$. This condition can be rewritten in the form

$$\left|\frac{dT}{dE}\right|F\omega\ll 1, \tag{4.5}$$

where $T=2\pi/\Omega$ is the period of the Duffing oscillator. The inequality (4.5) is violated in the vicinity of the saddle value of the energy $E=0$, which imposes the restriction on the field frequency. For example, if $F=1$, then for the 1:1 resonance, Eq. (4.5) gives the lower limit of the field frequency $\omega_-\sim 0.8$.

V. CONCLUSIONS

In this paper, a new characteristic of the response of the nonlinear systems to the external harmonic field, the harmonic susceptibility $A(\omega)$, has been introduced and studied. The harmonic susceptibility properly generalizes the common linear susceptibility but does not impose any restrictions on the parameters of the field or the character of the system's motion. The quantity $A(\omega)$ has the exact value if the states of the ensemble remain localized in the phase space; this condition is always fulfilled for the resonant case. In the chaotic case the overlap of resonances can produce the onset of the unlimited diffusion in the phase space, which eventually may lead to the decay of the system (e.g., the stochastic ionization of the highly excited atoms [11–14]). For this class of systems the harmonic susceptibility can serve as a useful characteristic of the transitional period.

The harmonic susceptibility has clear experimental meaning. If the ensemble of the identical noninteracting nonlinear oscillators with equal energies and random phases is subjected to the field of a monochromatic wave, then the intensity of the scattered radiation is proportional to the square of $A(\omega)$ provided that the spatial extent of the ensemble is small in comparison with the radiation wavelength.

We have studied only the one-dimensional example. However, the harmonic susceptibility can be calculated for systems with any number of degrees of freedom. That permits us to study the response of the system with the chaotic unperturbed motion in the harmonic fields of finite amplitude. At present, only the case of the infinitesimal field strength has been studied [15,16]. The concept of harmonic susceptibility can be straightforwardly generalized to the nonlinear harmonic susceptibilities that describe the response of the system on higher harmonics of the frequency of the perturbation in external harmonic fields of finite amplitudes.

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APPENDIX

The dependence of the characteristics of the synchronic component of the response on the field magnitude is an already known effect in quantum radiophysics. In particular, such a dependence occurs in the response of the two-level model [4].

The unperturbed system is described by the Hamiltonian $\hat{H}_0$ with the energy spectrum $\{E_n\}$ and wave functions $|n\rangle$. The homogeneous electric field of frequency ω and strength F/e, where e is the electron charge, is imposed along the OX axis. If $\omega \approx \omega_{mn} = (E_m - E_n)/\hbar$, $m > n$, then in the rotating-wave approximation the two-level system can be described by the equations for the components of its Bloch vector u, v, and w:

$$\dot{u} = -\Delta v, \quad \dot{v} = \Delta u + \Omega_r w, \quad \dot{w} = -\Omega_r v, \tag{A1}$$

where $\Delta = \omega - \omega_{mn}$, Ω_r is the Rabi frequency, $\Omega_r = 2 d_0 F/\hbar e$, and d_0 is the matrix element $d_0 = \langle n | e\hat{x} | m \rangle$. The component w describes the population inversion, whereas the components u and v are related to the atom's dipole moment $d(t)$:

$$d(t) = d_0 [u \cos \omega t - v \sin \omega t]. \tag{A2}$$

Solving Eq. (A1) with the initial conditions $u(0) = v(0) = 0$, $w(0) = -1$, which correspond to the initial state $\psi(0) = |n\rangle$, we obtain for the dipole moment

$$d(t) = \langle e\hat{x} \rangle = \frac{2 d_0^2 F}{\hbar e} \left[\frac{\Delta}{\Omega_s^2} \cos \omega t + \cdots \right], \tag{A3}$$

where Ω_s is the shifted Rabi frequency,

$$\Omega_s(\Delta) = \sqrt{\Omega_r^2 + \Delta^2}, \tag{A4}$$

and ''$\cdots$'' denotes terms with other frequencies. Thus, the harmonic susceptibility (1.8) of the two-level model is equal to

$$A(\omega) = \frac{2 d_0^2}{\hbar e^2} \frac{\Delta}{\Omega_s^2}. \tag{A5}$$

We note that in contrast with the properties of the harmonic susceptibility of the nonlinear oscillator, the HS of the two-level system decreases like $A(\omega) \propto F^{-2}$ in the limit of strong fields, and that $A(\omega) = 0$ in the exact resonance.

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